Homotopy:

Consider $h(t,x)=t\cdot f(x)+(1-t)\cdot f(x)$

Suppose one knows the solution of $g(x)=0$. We want to find the solution of $f(x)=0$ through solving a sequence of $h(t_i,x)=0,$ where $0=t_0<t_1<t_2<\cdots<t_n=1$.

The mapping $h:[0,1]\times X\to Y$ where $f,g:X\to Y$ is continuous and $h(0,x)=f(x)$ and $h(1,x)=g(x)$.

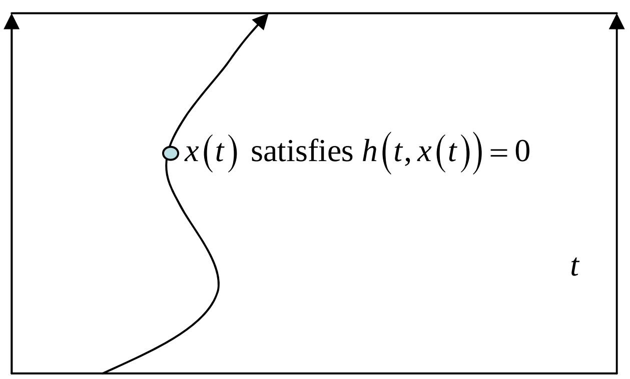
If such a mapping h exists, we call f and g are homotopic.

The method of finding zeros of $f(x)$ by the homotopy is called the continuation method.

Assume h is differentiable in t

consider $h(t,x(t))=0$ $\quad\begin{pmatrix}\text{here } x \text{ is the root of}\\ h_t(x)=h(t,x(t))=0\end{pmatrix}$

$h(1,x)=0\equiv$ roots of $g(x)=0$



$h(0,x)=0\equiv$ roots of $f(x)=0$

$\Rightarrow \frac{d}{dt}h(t,x(t))=0 \quad \Rightarrow \frac{\partial h}{\partial t}+\frac{\partial h}{\partial x}\cdot x'(t)=0$ $\left(\begin{array}{l}\text{true when } x'(t)\neq 0 \text{ by the} \\ \text{implicit function theorem}\end{array}\right)$

$\Rightarrow x'(t)=-h_x(t,x(t))^{-1}h_t(t,x(t)) \quad --(*)$ (Davideno ODE)

Since the differential equation has a known initial $x(0)=$ the root of $g(x)$, clearly,

$\Rightarrow x(t)=\int_0^t x'(t)dt+x(0)$

$\quad =\int_0^t -h_x^{-1}(t,x(\tau))\cdot h(t,x(\tau))d\tau$

In general, one can't compute the integration analytically.
One can solve the ODE numerically. To find the root of $h(t,x(t))=0$ at time t_k, one can employee the newton's iterations to find the numerical solution using x_k^0 as the initial where x_k^0 is a numerical solution of the Davideno ODE at time t_k.

Ex1. Let $g(x,y)=\begin{pmatrix} x^2-3y^2+3 \\ xy+6 \end{pmatrix}$. Find the zero of g.

Ans: Consider $f(x,y)=\begin{pmatrix} x^2-3y^2+2 \\ xy-1 \end{pmatrix}=0$

The solution of $f(x,y)$, $x=\pm 1$, y= ± 1 is easy to find.

Now condider the homotopy $h(t,x)=\begin{pmatrix} x^2-3y^2+2+t \\ xy-1+7t \end{pmatrix}$.

We have

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = -\begin{pmatrix} \frac{\partial}{\partial x}\left(x^2-3y^2+2+t\right) & \frac{\partial}{\partial y}\left(x^2-3y^2+2+t\right) \\ \frac{\partial}{\partial x}\left(xy-1+7t\right) & \frac{\partial}{\partial y}\left(xy-1+7t\right) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$= -\begin{pmatrix} 2x & -6y \\ y & x \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 7 \end{pmatrix} \quad -- \ (1)$$

with initial $\vec{x}_0 = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\vec{x}_0 = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

One can solve (1) numerically as following:

(i) $\frac{\Delta \vec{x}_n}{\Delta t} = -\begin{pmatrix} 2x_n & -6y_n \\ y_n & x_n \end{pmatrix}\begin{pmatrix} 1 \\ 7 \end{pmatrix}$

(ii) $\vec{x}_{n+1} = \vec{x}_n + \Delta \vec{x}_n$

(iii) Newton iterations on

$$h(x,y) = \begin{pmatrix} x^2-3y^2+2+(n+1)\Delta t \\ xy-1+7(n+1)\Delta t \end{pmatrix} = 0$$

using initial $\vec{x}_{n+1}$. Go to next time step when converges.

Major problems in continuation:

(1) $h_x^{-1}\left(\tau, x(\tau)\right)$ may not exist.

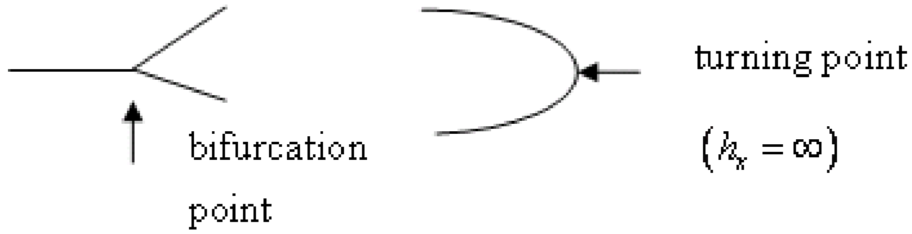
$h_x\left(\tau, x(\tau)\right) = 0 \Rightarrow x'(\tau)$ can be arbitrary

$\Rightarrow$ more than one possible update (bifurcation !)

(2) $h_x\left(\tau, x(\tau)\right) = \infty \Rightarrow h_x^{-1} = 0 \Rightarrow x'(\tau) = 0$

$\Rightarrow$ no update can be obtained from solving the ODE.

(meaning $x(t)$ turns around or has a limit point at time τ)



Review: continuation method

General problem:

Given a known solution $\left(\underset{(\text{vector})}{u_0} \; , \; \underset{(scalar)}{\lambda_0} \right)$ that satisfies $G\left(u_0, \lambda_0\right) = 0$

(1). Compute $\dfrac{du}{d\lambda}$ form

$$\frac{d}{d\lambda} G\left(u, \lambda\right) = 0 \quad \Rightarrow \quad \frac{\partial G}{\partial u} \cdot \frac{\partial u}{\partial \lambda} + \frac{\partial G}{\partial \lambda} = 0 \quad \Rightarrow \quad \underset{(matrix)}{G_u} \cdot \frac{du}{d\lambda} = - \underset{(vector)}{G_\lambda}$$

(2). Euler predictor step:

$$u^0 = u_0 + u_\lambda \left(\lambda - \lambda_0\right)$$

(3). Use u^0 as an initial guess in Newton's method

$G_u^i \left(u^{i+1} - u^i\right) = -G\left(u^i, \lambda\right)$, until converge.

(4).Use $\left(u\left(\lambda\right), \lambda\right)$ as $\left(u_0, \lambda_0\right)$ and go to (1)

Problems arise when G_u is singular

(i) if $G_\lambda \notin Range\left(G_u\right)$ then no solution for $\dfrac{du}{d\lambda}$

$\Rightarrow \left(u, \lambda\right)$ is called a limit point.

(ii) if $G_\lambda \in Range\left(G_u\right)$ then since G_u is singular,

more than one solution can be obtained for $\dfrac{du}{d\lambda}$

$\Rightarrow \left(u, \lambda\right)$ is called a bifucation point.

Arc-length method:

Insteat of parametrizing u by $\lambda(\text{or } t)$, we parametrize u by the arc-length s. In other word, we have

$$G(u(s),\lambda(s))=0 \quad \text{where } s=\int_0^t \sqrt{\left\|u^{'}(\tau)\right\|^2+\lambda^{'}(\tau)^2}\,d\tau$$

$$\Rightarrow \frac{d}{ds}G=0 \quad \Rightarrow G_u\dot{u}+G_\lambda\dot{\lambda}=0 \quad -(2)$$

$(\dot{u},\ \dot{\lambda}$ be the derivatives with respect to $s)$

Moreover, by the fact s is the arc-length at time t, we have

$$\frac{d}{ds}S=1=\frac{d}{dt}\int_0^t\sqrt{u^{'2}(\tau)+\lambda^{'}(\tau)}\,d\tau\cdot\frac{1}{\left(\frac{ds}{dt}\right)}$$

$$=\sqrt{\left\|\frac{du}{dt}\right\|^2+\left(\frac{d\lambda}{dt}\right)^2}\cdot\frac{1}{\left(\frac{ds}{dt}\right)}$$

$$=\sqrt{\left\|\frac{du}{ds}\right\|^2+\left(\frac{d\lambda}{ds}\right)^2}\cdot\left(\frac{ds}{dt}\right)\cdot\frac{1}{\left(\frac{ds}{dt}\right)}$$

$$\Rightarrow \|\dot{u}\|^2+\left(\dot{\lambda}\right)^2=1 \quad -(3)$$

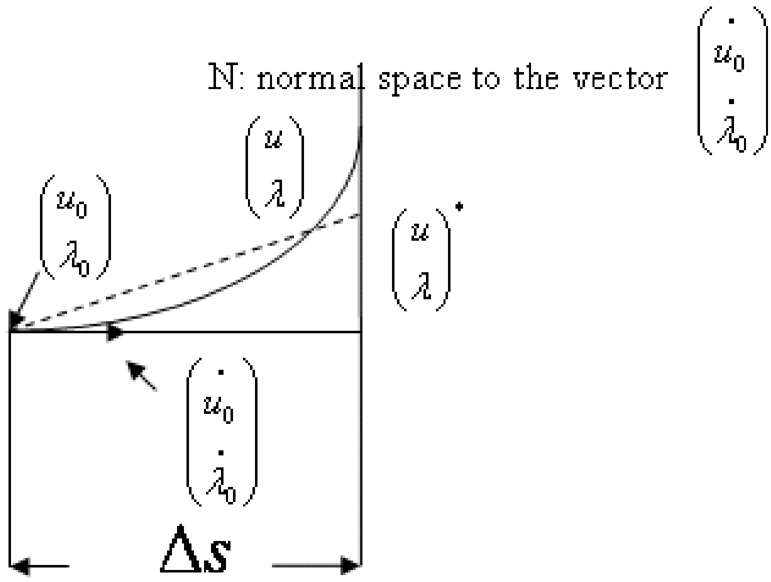
(2), (3) $\Rightarrow$ we now seek for solution $\begin{cases} G_u\dot{u}+G_\lambda\dot{\lambda}=0 \\ \|\dot{u}\|^2+\left|\dot{\lambda}\right|^2=1 \end{cases}$

(notice: $(\dot{u},\dot{\lambda})$ a unit vector)

$$\begin{cases} (G_u, G_\lambda)\begin{pmatrix} \dot{u} \\ \dot{\lambda} \end{pmatrix} = 0 \\ \dot{u}^T \cdot \dot{u} + \dot{\lambda}^T \cdot \dot{\lambda} = 1 \end{cases} \approx \begin{pmatrix} G_u(u_0, \lambda_0) & G_\lambda(u_0, \lambda_0) \\ \dot{u}_0{}^T & \dot{\lambda}_0{}^T \end{pmatrix}\begin{pmatrix} \Delta u \\ \Delta \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta s \end{pmatrix} \quad -(**)$$

- Forward Euler (one prediction step)

  we advance from s_0 to s along the tangent to the solution branch, the predicter solution (u, λ) has to satisfies $\begin{pmatrix} u \\ \lambda \end{pmatrix}^* =$ solution of $(**) + \begin{pmatrix} u_0 \\ \lambda_0 \end{pmatrix}$.



• Correction step:

Let $N\left(u(s),\lambda(s)\right) \equiv \dot{u}_0^{\ T}\left(u(s)-u(s_0)\right)+\dot{\lambda}_0\left(\lambda(s)-\lambda(s_0)\right)$

$$-\underbrace{\left(s-s_0\right)}_{\Delta s}=0-(4)$$

$\left(\begin{array}{l}\text{note: (4) is the linearization of (3) at } (\mathrm{u}_0,\lambda_0), N(u,\lambda)=0 \text{ is} \\ \qquad \text{the hyperplane perpendicular with } (\dot{\mathrm{u}}_0,\dot{\lambda}_0) \text{ and } \Delta s \\ \qquad \text{away from the } (\mathrm{u}_0,\lambda_0)\end{array}\right)$

We require the corrector solution remain on the hyperplane by

Solving $\begin{cases} G\left(u(s),\lambda(s)\right)=0 \\ N\left(u(s),\lambda(s)\right)=0 \end{cases}$ by chord iteration (or Newton iteration)

$$\Rightarrow \begin{pmatrix}\delta u^i \\ \delta\lambda^i\end{pmatrix} = -\underbrace{\begin{pmatrix} G_u & G_\lambda \\ N_u^T & N_\lambda \end{pmatrix}}_{A}^{-1} \cdot \begin{pmatrix} G \\ N \end{pmatrix}$$

$$= -\begin{pmatrix} G_u\left(u^i,\lambda^i\right) & G_\lambda\left(u^i,\lambda^i\right) \\ \dot{u}_0^{\ T} & \dot{\lambda}_0^{\ T} \end{pmatrix}^{-1} \begin{pmatrix} G\left(u^i,\lambda^i\right) \\ N\left(u^i,\lambda^i\right) \end{pmatrix}$$

until converge, here $\left(u^0,\lambda^0\right)=\left(u^*,\lambda^*\right)$.

Remark:

(1) A is nonsingular when $\underbrace{G_u \text{ is singular and } G_\lambda \notin \text{Range}\left(G_u\right)}_{\text{(limit point case)}}$

(2) The bifucation point case: $\left(G_u \text{ singular and } G_\lambda \in \text{Range}\left(G_u\right)\right)$

See H.B.Keller Numerical solution of bifurcation and nonlinear eigenvalue problems, in Application of Bifucation theory, P.Rabinowitz, ed, Academie Press, NewYork 1997 P.359-384 and the handouts.

(3) A free software package: Auto for computing Bifucation http://indy.cs.concordia.ca/auto

Example 0: $G(u,\lambda)=\begin{pmatrix}\lambda+u_1^2-u_2^2\\ u_1u_2\end{pmatrix}=0$

$$\Rightarrow G_u=\begin{pmatrix}2u_1 & 2u_2\\ u_2 & u_1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\ \text{ where } u_1=u_2=0$$

$$\Rightarrow G_\lambda=\begin{pmatrix}1\\0\end{pmatrix}\notin Range(G_u)\ \Rightarrow\left(\begin{pmatrix}0\\0\end{pmatrix},0\right)\text{ is a limit point.}$$

Example 1: considering $\frac{dy}{dt}=f(t,y),\ y(0)=y_0$

Let s be the arclength. We have

$$\begin{cases}\dfrac{dy}{ds}\cdot\dfrac{ds}{dt}=f(t,y)\\ \dfrac{dt}{ds}\cdot\dfrac{ds}{dt}=1\end{cases}$$

$$\Rightarrow\quad\left(\left(\frac{dt}{ds}\right)^2+\sum_{i=1}^{n}\left(\frac{dy_i}{ds}\right)^2\right)\left(\frac{ds}{dt}\right)^2=1+\sum_{i=1}^{n}f_i^2(t,y)$$

let $\left(\frac{dt}{ds}\right)^2+\sum_{i=1}^{n}\left(\frac{dy_i}{ds}\right)^2=1\Rightarrow\frac{ds}{dt}=\sqrt{1+\sum_{i=1}^{n}f_i^2}$

let $x=\begin{pmatrix}y\\t\end{pmatrix}$, we have

$$\frac{dx}{ds}=\frac{\begin{pmatrix}f(t,y)\\1\end{pmatrix}}{\sqrt{1+\sum_{i=1}^{n}f_i^2(t,y)}}\qquad ---(5)$$

Solve the Van der Pol's equation

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = \mu\left(1 - y_1^2\right) y_2 - y_1 \end{cases} \qquad -(6)$$

with $\begin{cases} y_1(0) = 2 \\ y_2(0) = 0 \end{cases}$ and μ=500

step1. change variables $\begin{cases} t = x\mu & z_1 = y_1 \\ \varepsilon = y\mu^2 & z_2 = \mu y_2 \end{cases}$

$$\Rightarrow \begin{cases} \dfrac{dz_1}{dx} = \dfrac{dy_1}{dt} \cdot \dfrac{dt}{dx} = y_2 \cdot \mu = z_2 \\ \dfrac{dz_2}{dx} = \dfrac{d(\mu y_1)}{dt} \cdot \dfrac{dt}{dx} = \mu^2 \left(\mu\left(1 - y_1^2\right) y_2 - y_1 \right) \\ \qquad = \mu^2 \left(\mu\left(1 - z_1^2\right) z_2 - z_1 \right) \end{cases}$$

The equations become

$$\begin{cases} \dfrac{dz_1}{dx} = z_2 \\ \varepsilon \dfrac{dz_2}{dx} = \left(1 - z_1\right)^2 z_2 - z_1 \end{cases} \qquad -(7)$$

Step2.

(i) Solve (6) by Runge-kutta mehtod.

classical 4th order Runge-kutta:

$$\begin{cases} y_{n+1} = y_n + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\ k_1 = f\left(x_n, y_n\right) \\ k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \\ k_3 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right) \\ k_4 = f\left(x_n + \frac{1}{2}h, y_n + hk_3\right) \end{cases}$$

(ii) Solve (7) by 4th Runge-kutta

(iii) Change (7) into the form (5)

$$\frac{d}{ds}\begin{pmatrix} z_1 \\ z_2 \\ x \end{pmatrix} = \frac{\begin{pmatrix} z_2 \\ \left[\left(1-z_1\right)^2 z_2 - z_1\right]\frac{1}{\varepsilon} \\ 1 \end{pmatrix}}{\sqrt{1+z_2^2+\left[\frac{\left(1-z_1\right)^2 z_2 - z_1}{\varepsilon}\right]^2}} \quad -(8)$$

$$\varepsilon = \frac{1}{250{,}000}.$$

Use Runge-kutta to solve (8)

(2)Use the arc-length method to solve the Bratu's Problem

$G(u,\lambda) = \Delta u + \lambda e^u = 0$ on Ω and $u\big|_{\Delta\Omega} = 0,\ \Omega = [0,1]$

find the limit points (u^*, λ^*)

Clearly, $G_u(u_0,\lambda_0) = \underbrace{\Delta + \lambda_0 e^u I}_{\text{matrix}}$

$$G_\lambda(u_0,\lambda_0) = \underbrace{e^{u_0}}_{\text{vector}}$$

Euler predict step:

$$\left(\text{let } u_0 \text{ be the trivial solution of } \begin{cases} \Delta u + 0\cdot e^u = 0 \\ u\big|_{\alpha\Omega} = 0 \end{cases} \right)$$

(1) $u^0 = u_0 + -G_u(u_0,0)^{-1} G_\lambda(u_0,0)\cdot(\delta\lambda)$

(2) ues $(u^0, \delta\lambda)$ as initial guess

$$\delta u^i = -\left(G_u^i\right)^{-1} G\left(u^i,\lambda\right);\ u^{i+1} = u^i + \delta u^i$$

converge $\Rightarrow (u_1,\lambda_1)$ here $\lambda_1 = \delta\lambda$

(3) $\delta s = \sqrt{\|u_1 - u_0\|^2 + |\Delta\lambda|^2} \quad (\delta u_1 = u_1 - u_0)$

$$\dot{u}_1 = \frac{\delta u_1}{\delta s};\ \dot{\lambda}_1 = \frac{\delta\lambda}{\delta s}$$

start arc-length iteration with fixed length δs

(一) Solve $\begin{pmatrix} G_u(u_1,\lambda_1) & G_\lambda(u_1,\lambda_1) \\ \dot{u}_1^T & \dot{\lambda}_1^T \end{pmatrix} \begin{pmatrix} \delta u \\ \delta\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta s \end{pmatrix}$

(二) $\begin{pmatrix} u \\ \lambda \end{pmatrix}^* = \begin{pmatrix} u_1 \\ \lambda_1 \end{pmatrix} + \begin{pmatrix} \delta u \\ \delta\lambda \end{pmatrix}$ and

$$N(u,\lambda) = \dot{u}_1^T (u - u_1) + \dot{\lambda}_1^T (\lambda - \lambda_1) - \Delta s$$

(三) Newton (Chord): $(u^0, \lambda^0) = (u^*, \lambda^*)$

$$\begin{pmatrix} \delta u \\ \delta\lambda \end{pmatrix} = -\begin{pmatrix} G_u(u^*,\lambda^*) & G_\lambda(u^*,\lambda^*) \\ \dot{u}_1^T & \dot{\lambda}_1^T \end{pmatrix}^{-1} \begin{pmatrix} G(u^i,\lambda^i) \\ N(u^i,\lambda^i) \end{pmatrix}$$

and $\begin{matrix} u^{i+1} = u^i + \delta u \\ \lambda^{i+1} = \lambda^i + \delta\lambda \end{matrix}$ until converge $\Rightarrow (\mathrm{u}_2, \lambda_2)$.

Let u_2, λ_2 and compute $\dot{u}_2 = \dfrac{u_2 - u_1}{\delta s}, \dot{\lambda}_2 = \dfrac{\lambda_2 - \lambda_1}{\delta s}$,

$$\delta s = \sqrt{\|u_2 - u_1\|^2 + \|\lambda_2 - \lambda_1\|}$$

(四) Go for next arc-length step. (repeat (一) ~ (三))

For the bifurcation case:

Consider $G(u,\lambda)=0$ let $x=(u,\lambda)$

$$G(x)=0 \overset{\frac{d}{ds}}{\Rightarrow} G_x\left(x(s)\right)\cdot \dot{x}(s)=0$$

$$\overset{\frac{d^2}{ds}}{\Rightarrow} G_{xx}\left(x(s)\right)\dot{x}(s)\dot{x}(s)+G_x\left(x(s)\right)\cdot x(s)=0 \quad -(***)$$

consider $\dot{x}_0=\alpha\phi_1+\beta\phi_2$ for some $\alpha,\beta\in\mathbb{R},$ here

$\phi_1,\phi_2\in Null\left(G_x\right)$ and $\psi\in Null\left(G_x^*\right)\left(\text{i.e } \left(\psi^* G_x\right)=0\right)$

(***) can be written as

$$\underbrace{\left(\psi^* G_{xx}\phi_1\phi_1\right)}_{c_{11}}\alpha+\underbrace{\left(\psi^* G_{xx}\phi_1\phi_2+\psi^* G_{xx}\phi_2\phi_1\right)}_{2c_{12}}\alpha\beta+\underbrace{\psi^* G_{xx}\phi_2\phi_2}_{c_{22}}\beta^2=0 \quad -(+++)$$

When the discriminant

$$\Delta\left(x^0\right)=\Delta\left(u^0,\lambda^0\right)=c_{12}^2-c_{11}c_{22}>0,$$

(+++) has 2 distinct real solutions.

$\Rightarrow$ $\left(u^0,\lambda^0\right)$ is a bifucation point with 2 solution brandies passing through it.

Example 3: consider the 2-species predator-prey model

$$\begin{cases} u_1' = 3u_1\left(1-u_1\right)-u_1u_2-\lambda\left(1-e^{-5u_1}\right) \\ u_2' = -u_2+3u_1u_2 \end{cases}$$

$$G_x=\left(G_{u_1},G_{u_2},G_\lambda\right)$$

$$=\begin{bmatrix} 3-6u_1-u_2-5\lambda e^{-5u_1}-\left(\frac{d}{dt}\right) & -u_1 & -\left(1-e^{-5u_1}\right) \\ 3u_2 & -1+3u_1-\left(\frac{d}{dt}\right) & 0 \end{bmatrix}$$

$$G_{xx}=\begin{bmatrix} \left(-6+25\lambda e^{-5u_1},-1,-5e^{-5u_1}\right) & \left(-1,0,0\right) & \left(-5e^{-5u_1},0,0\right) \\ \left(0,3,0\right) & \left(3,0,0\right) & \left(0,0,0\right) \end{bmatrix}$$

For $u_1=u_2=0,\ \lambda=\frac{3}{5},$ one has

$$G_x^0=\begin{pmatrix} -\frac{d}{dt} & 0 & 0 \\ 0 & -1-\frac{d}{dt} & 0 \end{pmatrix}\Rightarrow Null\left(G_x^0\right)=span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$\left(G_x^0\right)^*=\begin{pmatrix} -\frac{d}{dt} & 0 \\ 0 & -1-\frac{d}{dt} \\ 0 & 0 \end{pmatrix}\Rightarrow Null\left(G_x^0\right)^*=span\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$$

Let $\phi_1=\begin{pmatrix}1\\0\\0\end{pmatrix},\ \phi_2=\begin{pmatrix}0\\0\\1\end{pmatrix}$ and $\psi=\begin{pmatrix}1\\0\end{pmatrix}$

$$G_{xx}^0=\begin{pmatrix} \left(9,-1,-5\right) & \left(-1,0,0\right) & \left(-5,0,0\right) \\ \left(0,3,0\right) & \left(3,0,0\right) & \left(0,0,0\right) \end{pmatrix}$$

$$G_{xx}^0\phi_1=\begin{pmatrix} 9 & -1 & -5 \\ 0 & 3 & 0 \end{pmatrix};\ G_{xx}^0\phi_2=\begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have

$$c_{11} = \psi^* G^0_{xx}\phi_1\phi_1 = (1,0)\begin{pmatrix} 9 & -1 & -5 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 9$$

$$2c_{12} = \quad \psi^* G^0_{xx}\phi_1\phi_2 = (1,0)\begin{pmatrix} 9 & -1 & -5 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -5$$

$$+\psi^* G^0_{xx}\phi_2\phi_1 = (1,0)\begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -5$$

$$c_{22} = \psi^* G^0_{xx}\phi_2\phi_2 = (1,0)\begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$\Rightarrow \Delta = (-5)^2 - 4\cdot 9\cdot 0 > 0 \ \Rightarrow \left(0, 0, \frac{3}{5}\right)$ is a bifucation point.

$(+++) \Rightarrow 9\alpha^2 - 10\alpha\beta = 0 \Rightarrow \alpha(9\alpha - \beta) = 0$

$\Rightarrow$ There are 2 solutions $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \dot{x} = 0\phi_1 + 1\phi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ \Rightarrow \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

or $\dot{x} = 10\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 9\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 9 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 9 \end{pmatrix}$

Remaek: To determine the bifucation points, one needs to find

$$\underbrace{Null\left(G_x\right)}_{\phi_1,\phi_2} \text{ and } \underbrace{Null\left(G_x^*\right)}_{\psi} \text{ in order to determine } c_{11}, c_{12}, c_{22}$$

Suppose ϕ_1 is a given branch $\left(\begin{array}{l}\text{we have it from following the}\\ \text{solution path}\end{array}\right)$

$$\Rightarrow \phi_1 = \dot{x}_0 = \alpha_1\phi_1 + \beta_1\phi_2 \Rightarrow \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \text{ must be a root of } (+++)$$

$$c_{11}(1) + 2c_{12}\cdot 1\cdot 0 + c_{22}\cdot 0 = 0 \;\Rightarrow c_{11} = 0$$

As a result, the second root must satisfy $2c_{12}\alpha + c_{22}\beta = 0$

$$\Rightarrow \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}c_{22}\\ -2c_{12}\end{pmatrix} = \begin{pmatrix}\psi^* G_{xx}\phi_2\phi_2\\ -2\psi^* G_{xx}\phi_2\phi_1\end{pmatrix}$$

Suppose one choose $\phi_2 \perp \phi_1 \Rightarrow \phi_2 \in Null\begin{pmatrix}G_x(x_0)\\ \phi_1\end{pmatrix} = Null\begin{pmatrix}G_x(x_0)\\ \dot{x}_0^T\end{pmatrix}$

and $\psi \in Null\left(G_x^T(x_0), \dot{x}_0\right)$.

Branch swithcing: find x_1, ϕ_2 by Newton iteration on the following system equations.

$$\begin{cases}\begin{cases} G(x_1) = 0\\ (x_1 - x_0)\cdot\phi_2\cdot\Delta s = 0\end{cases}\\ \begin{pmatrix}G_x(x_0)\\ \dot{x}_0^T\end{pmatrix}\phi_2 = 0;\\ \phi_2^T\phi_2 = 1\end{cases}$$